

# FOURIER MULTIPLIERS FOR HARDY SPACES OF DIRICHLET SERIES

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**ABSTRACT.** We obtain new results on Fourier multipliers for Dirichlet-Hardy spaces. As a consequence, we establish a Littlewood-Paley type inequality which yields a simple proof that the Dirichlet monomials form a Schauder basis for  $p > 1$ .

## 1. INTRODUCTION

The Dirichlet-Hardy spaces  $\mathcal{H}^p$  were first explicitly studied in the papers [2, 8]. (We refer to these papers for full details of the discussion in this section. See also [6] for some historical remarks.) For  $p = 2$ , they consist of Dirichlet series  $\sum_{n \in \mathbb{N}} a_n n^{-s}$  with square-summable coefficients, where  $s = \sigma + it$  denotes the complex variable. By the Cauchy-Schwarz inequality, functions in  $\mathcal{H}^2$  converge on the half-plane  $\mathbb{C}_{1/2} = \{\sigma > 1/2\}$ . These spaces connect function space theory to analytic number theory. A striking illustration of this connection is given by the Riemann-zeta function  $\zeta(s) = \sum_{n \in \mathbb{N}} n^{-s}$  that gives the reproducing kernel of  $\mathcal{H}^2$ . Indeed, the function  $k_w(s) := \zeta(s + \bar{w})$ , for  $\operatorname{Re} w > 1/2$ , has the property that  $\langle f | k_w \rangle = f(w)$  for all  $f \in \mathcal{H}^2$ , as may be verified by inspection.

For general  $p > 0$ , these spaces are defined to be the closure of Dirichlet polynomials in the norm

$$\lim_{T \rightarrow \infty} \left( \frac{1}{2T} \int_{-T}^T \left| \sum_{n=1}^N a_n n^{-it} \right|^p dt \right)^{1/p}. \quad (1)$$

This norm can be understood as the ergodic theorem on the infinite dimensional polydisk  $\mathbb{T}^\infty$ . To briefly explain this, we note that  $\mathbb{T}^\infty$  is a compact Abelian group with the product of the normalised Lebesgue measures  $d\theta_i/2\pi$  on each copy of  $\mathbb{T}$  as its unique normalised Haar measure  $d\theta$ . It has dual group  $\mathbb{Z}_{\text{fin}}^\infty$ , i.e., sequences in  $\mathbb{Z}^\infty$  with finitely many non-zero coefficients. So by standard Fourier analysis on groups,  $F \in L^p(\mathbb{T}^\infty)$  has a Fourier expansion  $F \sim \sum_{\nu \in \mathbb{Z}_{\text{fin}}^\infty} a_\nu z^\nu$ , where  $z \in \mathbb{T}^\infty$  and we use multi-index notation. The central observation, essentially

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called Kroenecker's lemma, is that the path  $\phi : t \mapsto (2^{-it}, \dots, p_i^{-it}, \dots)$ , where  $p_i$  is the  $i$ 'th prime number, is ergodic in  $\mathbb{T}^\infty = \{z = (z_1, \dots) : z_i \in \mathbb{T}\}$ . The ergodic theorem now says exactly that for continuous functions

$$\lim_{T \rightarrow \infty} \left( \frac{1}{2T} \int_{-T}^T |F \circ \phi(t)|^p dt \right)^{1/p} = \|F\|_{L^p(\mathbb{T}^\infty)} \quad (2)$$

For  $F$  with spectral support only in the narrow cone  $\mathbb{N}_{\text{fin}}^\infty$ , one checks that  $F \circ \phi$  is a Dirichlet series and that the right-hand side of this formula is exactly (1), provided we identify  $a_\nu$  with  $a_n$  when  $n = p_1^{\nu_1} p_2^{\nu_2} \dots$ . (Note that the same argument can be made using only the Stone-Weierstrass theorem, see [11]) We define the subspace  $H^p(\mathbb{T}^\infty)$  to consist of exactly these functions. By the uniqueness of prime number factorization, the map from  $H^p(\mathbb{T}^\infty)$  to  $\mathcal{H}^p$  given by  $F \mapsto F \circ \phi$  has an inverse, which is called the Bohr lift in honor of H. Bohr.

The structure of the paper is as follows. In Section 2, we use a technique of Fefferman to study certain Fourier multipliers on the spaces  $L^p(\mathbb{T}^\infty)$ . These results are used in Section 3 to obtain a Littlewood-Paley inequality for the spaces  $\mathcal{H}^p$ : for  $f = \sum_{n \in \mathbb{N}} a_n n^{-s}$  in  $\mathcal{H}^p$  with  $p > 1$  and  $c > 1$ , we have

$$\|f\|_{\mathcal{H}^p} \simeq |a_0| + \left\| \left( \sum_{k \geq 0} \left| \sum_{\log n \in (c^k, c^{k+1})} a_n n^{-s} \right|^2 \right)^{1/2} \right\|_{\mathcal{H}^p}. \quad (3)$$

As an application, we observe that the functions  $\{n^{-s}\}_{n \in \mathbb{N}}$  constitute a Schauder basis for the spaces  $\mathcal{H}^p$  for  $p > 1$ .

## 2. FOURIER MULTIPLIERS

To state and prove our theorem on Fourier multipliers, we first introduce some notation, and review some necessary background. Throughout the section,  $p \geq 1$ .

A measurable function  $m : \mathbb{R} \rightarrow \mathbb{C}$  is called a Fourier multiplier on  $L^p(\mathbb{R})$  if the operator  $f \mapsto \mathcal{F}^{-1}(m(\xi)\hat{f}(\xi))$  is bounded on  $L^p(\mathbb{R})$ , where  $\mathcal{F}$  denotes the Fourier transform. On the torus  $\mathbb{T}$ , a function  $m : \mathbb{Z} \rightarrow \mathbb{C}$  is called a multiplier if the map defined by the relation  $e^{int} \mapsto m(n)e^{int}$  extends to a bounded operator on  $L^p(\mathbb{T})$ . Finally, a function  $m : \mathbb{Z}_{\text{fin}}^\infty \rightarrow \mathbb{C}$  is called a multiplier if the operator

$$\sum_{\nu \in \mathbb{Z}_{\text{fin}}^\infty} a_\nu e^{i\nu \cdot \theta} \mapsto \sum_{\nu \in \mathbb{Z}_{\text{fin}}^\infty} m(\nu) a_\nu e^{i\nu \cdot \theta}$$

is bounded on  $L^p(\mathbb{T}^\infty)$ . Here we use the notation  $z = e^{i\theta}$  for a point in  $\mathbb{T}^\infty$ . We denote the respective operator norms by  $\|m\|_{M_p(X)}$ , where  $X = \mathbb{R}, \mathbb{T}$  or  $\mathbb{T}^\infty$  as appropriate. We refer to the operator of multiplication by  $m$  by  $T_m$ .

It is well-known that results for multipliers on  $\mathbb{T}$  may be deduced from those on the real line by the method of transference. More specifically, let  $m : \mathbb{R} \rightarrow \mathbb{C}$  be a

regulated function, i.e.,

$$m(\xi) = \lim_{\epsilon \rightarrow 0^+} \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} m(\xi + t) dt, \quad \forall \xi \in \mathbb{R}.$$

The basic result on transference, due to de Leeuw [4] (see [5, Section 3.6.2] for proofs), states that if a regulated function  $m$  is a multiplier on  $\mathbb{R}$ , then  $m$  restricted to  $\mathbb{Z}$  is a multiplier on the torus. A converse statement also holds. In fact,

$$\|m\|_{M_p(\mathbb{R})} = \sup_{\gamma > 0} \|m(\gamma \cdot)\|_{M_p(\mathbb{T})}. \quad (4)$$

Our argument relies in a crucial way on this formula.

To formulate our result, we introduce some additional notation. For  $\nu \in \mathbb{Z}_{\text{fin}}^\infty$ , one associates a unique rational number:

$$r : \nu \mapsto r_\nu = p_1^{\nu_1} \cdots p_k^{\nu_k},$$

where  $p_i$  is the  $i$ 'th prime number. So, given a function  $m : \mathbb{Q}_+ \rightarrow \mathbb{C}$ , we obtain a function  $m \circ r : \mathbb{Z}_{\text{fin}}^\infty \rightarrow \mathbb{C}$ . In particular,  $m$  induces in this way a densely defined Fourier multiplier on  $L^p(\mathbb{T}^\infty)$  by

$$T_{m \circ r} : \sum_{\nu \in \mathbb{Z}_{\text{fin}}^\infty} a_\nu e^{i\nu \cdot \theta} \mapsto \sum_{\nu \in \mathbb{Z}_{\text{fin}}^\infty} m(r_\nu) a_\nu e^{i\nu \cdot \theta}$$

Our multiplier result is as follows:

**Theorem 1.** *Let  $p \in [1, \infty)$  and  $m : \mathbb{R}_+ \rightarrow \mathbb{C}$  be a regulated function continuous at rational points. Then  $m \circ r$  is a Fourier multiplier on  $L^p(\mathbb{T}^\infty)$ , where  $r(\nu) = p_1^{\nu_1} \cdots p_k^{\nu_k}$  for  $\nu \in \mathbb{Z}_{\text{fin}}^\infty$ , if and only if  $m \circ \exp$  is a Fourier multiplier on  $L^p(\mathbb{R})$ . Moreover,*

$$\|m \circ r\|_{M_p(\mathbb{T}^\infty)} = \|m \circ \exp\|_{M_p(\mathbb{R})}.$$

*Proof.* We split the proof of the theorem into two parts.

First, we establish that  $\|m \circ r\|_{M_p(\mathbb{T}^\infty)} \leq \|m \circ \exp\|_{M_p(\mathbb{R})}$ . Fix a polynomial

$$f = \sum_{\nu \in \mathbb{Z}_{\text{fin}}^\infty} a_\nu e^{i\nu \cdot \theta}.$$

Observe that since a polynomial only depends on a finite number of variables, we may restrict our attention to  $L^p(\mathbb{T}^d)$ , for some  $d \in \mathbb{N}$ . As a multiplier on  $L^p(\mathbb{T}^d)$ , we need only consider  $\nu \in \mathbb{Z}^d$ . Explicitly, we only need to consider the multiplier

$$\nu \mapsto m(r_\nu) = m(e^{\nu_1 \log p_1 + \dots + \nu_d \log p_d}), \quad \nu \in \mathbb{Z}^d,$$

acting on  $L^p(\mathbb{T}^d)$ . The idea is to introduce a change of variables on  $\mathbb{T}^d$  so that as a multiplier, this function only acts on the first variable.

To do this, we need to make an approximation. For  $\delta > 0$ , choose  $Q, a_1, \dots, a_d \in \mathbb{N}$  so that

$$\left| \frac{a_j}{Q} - \log p_j \right| < \delta, \quad \text{for } j = 1, \dots, d.$$

We may assume that  $a_1$  and  $a_2$  are relatively prime (indeed, by the prime number theorem, we may choose both  $a_1$  and  $a_2$  to be prime), whence there exist  $q_1, q_2 \in \mathbb{N}$  so that  $a_1 q_2 - a_2 q_1 = 1$ . This ensures that the  $d \times d$  matrix

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_d \\ q_1 & q_2 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

satisfies  $\det A = 1$ . A fortiori,  $A^{-1}$  also has integer coefficients, whence  $A : \mathbb{Z}^d \rightarrow \mathbb{Z}^d$  is bijective. Especially, one checks that  $A$  induces a bijective and measure preserving diffeomorphism on  $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$ .

We next introduce a function defined on  $\nu \in \mathbb{Z}^d$  by

$$M(\nu) = m \left( e^{\frac{a_1}{Q}\nu_1 + \cdots + \frac{a_d}{Q}\nu_d} \right).$$

Since  $m$  was assumed to be continuous on rational numbers, it follows that for any  $\epsilon > 0$  small enough, we may choose  $\delta > 0$  sufficiently small in the above approximation, so that  $|M(\nu) - m(r_\nu)| < \epsilon$  uniformly on the finite index set corresponding to the set of non-zero coefficients of the polynomial  $f$ . In particular, this implies that we can make  $\|T_M f - T_{m \circ r} f\|_{L^p(\mathbb{T}^d)}$  arbitrarily small. In light of (4), to obtain the desired inequality, we infer that it suffices to prove

$$\|T_M f\|_{L^p(\mathbb{T}^d)} \leq \|m \circ \exp(Q^{-1} \cdot)\|_{M_p(\mathbb{T})}. \quad (5)$$

To verify (5), let us first employ the change of variables  $\theta = A^T \theta'$  to get

$$\|T_M f\|_{L^p(\mathbb{T}^d)}^p = \int_{\mathbb{T}^d} \left| \sum_{\nu \in \mathbb{Z}^d} M(\nu) a_\nu e^{i\nu \cdot A^T \theta'} \right|^p d\theta' = \int_{\mathbb{T}^d} \left| \sum_{\nu \in \mathbb{Z}^d} M(\nu) a_\nu e^{iA\nu \cdot \theta'} \right|^p d\theta'.$$

If we change indices by  $\nu' = A\nu$ , and observe that  $M(\nu) = m(e^{\nu'_1/Q})$ , this becomes

$$\int_{\mathbb{T}^d} \left| \sum_{\nu' \in \mathbb{Z}^d} m(e^{\nu'_1/Q}) a_{A^{-1}\nu'} e^{i\nu' \cdot \theta'} \right|^p d\theta' = \int_{\mathbb{T}^{d-1}} \left\| \sum_{\nu'_1} b_{\nu'_1} m(e^{\nu'_1/Q}) e^{i\nu'_1 \theta'_1} \right\|_{L^p(d\theta'_1)}^p \frac{d\theta'_2}{2\pi} \cdots \frac{d\theta'_d}{2\pi}, \quad (6)$$

where  $b_{\nu'_1} = b_{\nu'_1}(\theta'_2, \dots, \theta'_d)$  is constant with respect to  $\theta'_1$ . This is less than or equal to

$$\begin{aligned} \|m \circ \exp(Q^{-1} \cdot)\|_{M_p(\mathbb{T})}^p \int_{\mathbb{T}^{d-1}} \left\| \sum_{\nu'_1} b_{\nu'_1} e^{i\nu'_1 \theta'_1} \right\|_{L^p(d\theta'_1)}^p \frac{d\theta'_2}{2\pi} \cdots \frac{d\theta'_d}{2\pi} \\ = \|m \circ \exp(Q^{-1} \cdot)\|_{M_p(\mathbb{T})}^p \|f\|_{L^p(\mathbb{T}^d)}^p, \end{aligned}$$

which exactly yields the desired inequality (5).

We turn to the second part of the proof, where we establish the inequality  $\|m \circ \exp\|_{M_p(\mathbb{R})} \leq \|m \circ r\|_{M_p(\mathbb{T}^\infty)}$ . By (4), it is sufficient to show that, for every  $\gamma > 0$ , we have  $\|m \circ \exp(\gamma \cdot)\|_{M_p(\mathbb{T})} \leq \|m \circ r\|_{M_p(\mathbb{T}^\infty)}$ . We now fix a polynomial

in one variable. As our idea is to work the previous argument backwards using only two variables, we express the polynomial as trivially depending on a second variable:

$$f(\theta'_1, \theta'_2) = \sum_{|n| \leq N} a_{(n,0)} e^{in\theta'_1}.$$

Here  $a_{(n,m)}$  is zero for all  $(n, m) \notin \mathbb{N} \times \{0\}$ .

As in the first part of the proof, we first fix  $\delta > 0$  and introduce a change of variables, this time induced by the matrix

$$B = \begin{pmatrix} b+1 & b \\ 1 & 1 \end{pmatrix}.$$

Above, the integer  $b$  is chosen so large that there exist prime numbers  $p_j, p_k$  for which

$$|\gamma(b+1) - \log p_j| < \delta/N, \quad \text{and} \quad |\gamma b - \log p_k| < \delta/N.$$

This is possible, since from the prime number theorem it holds that  $\log(p_{n+1}/p_n) \rightarrow 0$  when  $n \rightarrow \infty$ . As we have  $\det B = 1$ , the matrix  $B$  induces a measure preserving diffeomorphism of  $\mathbb{T}^2$ .

Setting  $\theta = B^T \theta'$  and  $(n, 0)^T = B\nu$ , we get

$$\begin{aligned} \|T_{m \circ \exp(\gamma \cdot)} f\|_{L^p(\mathbb{T})}^p &= \iint_{\mathbb{T}^2} \left| \sum_{|n| \leq N} m(e^{\gamma n}) a_{(n,0)} e^{i(n,0) \cdot \theta'} \right|^p \frac{d\theta'_1}{2\pi} \frac{d\theta'_2}{2\pi} \\ &= \iint_{\mathbb{T}^2} \left| \sum_{\nu \in \mathbb{Z}^2} m(\exp(\gamma((b+1)\nu_1 + b\nu_2))) a_{B\nu} e^{i\nu \cdot \theta} \right|^p \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi}. \end{aligned}$$

As  $v = (n, -n)^T$ , we are only summing over  $\nu \in \mathbb{Z}^2$  for which  $|\nu| \leq 2N$ . So given any  $\epsilon > 0$ , by choosing  $\delta > 0$  small enough, we make

$$|m \circ \exp(\gamma n) - m(p_j^{\nu_1} p_k^{\nu_2})| = |m(e^{\gamma(b_1 \nu_1 + b_2 \nu_2)}) - m(e^{\nu_1 \log p_j + \nu_2 \log p_k})| < \epsilon$$

uniformly for indices  $\nu$  so that  $a_{B\nu}$  is non-zero. This implies that we only need to establish that

$$\int_{\mathbb{T}^2} \left| \sum_{\nu \in \mathbb{Z}^2} m(p_j^{\nu_1} p_k^{\nu_2}) a_{B\nu} e^{i\nu \cdot \theta} \right|^p \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} \leq \|m \circ r\|_{M_p(\mathbb{T}^\infty)}^p \|f\|_{L^p(\mathbb{T})}^p.$$

But this is readily seen to hold, as the left-hand side may be interpreted as  $T_{m \circ r}$  applied to a function  $F$  depending on the  $j$ 'th and  $k$ 'th copy of  $\mathbb{T}$  in  $\mathbb{T}^\infty$ , and where  $\|F\|_{L^p(\mathbb{T}^\infty)} = \|f\|_{L^p(\mathbb{T})}$  holds by reversing the changes in notation and variables.  $\square$

### 3. SOME CONSEQUENCES AND OPEN PROBLEMS

In this section we deduce a Littlewood-Paley inequality from Theorem 1, and also discuss Schauder bases for the spaces  $\mathcal{H}^p$ .

First, we observe how a characterisation due to Marcinkiewicz is inherited by multipliers of the form discussed in the previous section. To do this, we recall that the total variation of a complex function  $f$  on the interval  $(a, b)$  is given by

$$\|f\|_{\text{BV}(a,b)} = \sup \sum_{n=1}^N |f(x_n) - f(x_{n-1})|,$$

where the supremum is taken over all sequences  $a = x_0 < x_1 < \dots < x_n = b$ . For fixed  $\eta > 1$ , we also use the notation

$$I_k = \begin{cases} [e^{\eta^k}, e^{\eta^{k+1}}] & k \geq 1. \\ [e^{-\eta}, e^{\eta}] & k = 0, \\ [e^{-\eta^{|k|+1}}, e^{-\eta^{|k|}}] & k \leq -1. \end{cases}$$

We now get:

**Corollary 1.** *Suppose that  $p \in (1, \infty)$  and  $\eta > 1$ , then there exists a constant  $C > 0$  such that for all regulated  $m : \mathbb{R}_+ \rightarrow \mathbb{C}$  that are continuous at rationals we have*

$$\|m \circ r\|_{M_p(\mathbb{T}^\infty)} \leq C \left( \|m\|_{L^\infty(0,\infty)} + \sup_{k \in \mathbb{Z}} \|m\|_{\text{BV}(I_k)} \right).$$

*Proof.* Since  $m \circ \exp$  and  $m$  have the same sup-norm, and  $\|m \circ \exp\|_{\text{BV}(\eta^k, \eta^{k+1})} = \|m\|_{\text{BV}(I_k)}$ , the Marcinkiewicz bound follows immediately from its classical counterpart, see [5, Section 5.2.1].  $\square$

We also formulate a Hörmander-Mihlin type multiplier theorem for  $p = 1$ , see [9, 7] (a proof is also found in [5, Theorem 5.2.7]). Recall that  $m : \mathbb{R} \rightarrow \mathbb{C}$  satisfies the Hörmander-Mihlin condition if  $m$  is continuous and piecewise differentiable on  $\mathbb{R} \setminus \{0\}$  with

$$\|m\|_{L^\infty(\mathbb{R})} + \sup_{x \neq 0} |xf'(x)| < \infty. \quad (7)$$

If this holds, then  $m \in M^p(\mathbb{R})$  for any  $p \in (1, \infty)$ . In addition,  $m$  defines a multiplier operator that is bounded from  $H^1(\mathbb{R})$  to  $L^1(\mathbb{R})$  with norm bounded by (7). For our purposes it is useful to observe that (7) remains invariant if  $m$  is replaced by  $m(\lambda \cdot)$  for any  $\lambda > 0$ .

**Corollary 2.** *Assume that  $m : (0, \infty) \rightarrow \mathbb{C}$  is continuous and piecewise differentiable. Then*

$$\|m \circ r\|_{H^1(\mathbb{T}^\infty) \rightarrow L^1(T^\infty)} \leq c \left( \|m\|_{L^\infty(0,\infty)} + \sup_{t > e} |t \log(t) m'(t)| \right).$$

*Proof.* The condition of  $m$  ensures that it can be modified on  $(0, e)$  so that it satisfies (7) on  $\mathbb{R}$ . Hence  $T_{m \circ \exp} : H^1 \rightarrow L^1$  is bounded in one variable with the stated bound. The case  $p = 1$  of the proof of Theorem 1 now applies without changes. One simply needs to observe that after the change of variables, the assumption  $f \in H^1(\mathbb{T}^\infty)$  implies that  $a_{A^{-1}\nu'} = 0$  if  $\nu_1 < 0$ , whence the one-dimensional multiplier  $m(e^{\nu'_1/Q})$  is applied only to analytic functions.  $\square$

We proceed to obtain a Paley-Littlewood type of theorem for  $L^p(\mathbb{T}^\infty)$  as a consequence of Corollary 1. Fix a rational number  $\eta > 1$ , and consider intervals  $I_k$  as above. For  $f = \sum_{\nu \in \mathbb{Z}_{\text{fin}}^\infty} a_\nu e^{i\nu \cdot \theta}$  in  $L^p(\mathbb{T}^\infty)$ , define the square function

$$S(f) = \left( \sum_k |f_k(\theta)|^2 \right)^{1/2},$$

where

$$f_k(\theta) = \sum_{\nu: r_\nu \in I_k} a_\nu e^{i\nu \cdot \theta}.$$

The following result is clearly the most interesting in the special case of  $\mathcal{H}^p$ , which we stated as formula (3) in the introduction.

**Corollary 3.** *Suppose that  $p \in (1, \infty)$ , and that  $\eta > 1$  is a rational number. Then there exist constants such that for all  $f \in L^p(\mathbb{T}^\infty)$ , we have*

$$\|f\|_{L^p(\mathbb{T}^\infty)} \simeq \|S(f)\|_{L^p(\mathbb{T}^\infty)}.$$

*Proof.* We apply a standard argument. Define  $m_\epsilon = \sum_{k \in \mathbb{Z}} \epsilon_k \chi_{I_k}$  for given  $\epsilon \in \{-1, 1\}^\mathbb{Z}$ . By Corollary 1, there exists some  $C > 0$ , independent of  $\epsilon$ , such that  $\|m_\epsilon \circ r\|_{M_p(\mathbb{T}^\infty)} \leq C$ . Here,  $m_\epsilon$  is made regulated by defining it appropriately on the endpoints of the intervals  $I_k$ . This has no effect on the operator  $T_{m_\epsilon \circ r}$  as the endpoints are irrational. Next, since  $T_{m_\epsilon \circ r} T_{m_\epsilon \circ r} = \text{Id}$ , we obtain for any  $g \in L^p(\mathbb{T}^\infty)$  that  $\|T_{m_\epsilon \circ r} g\|_{L^p(\mathbb{T}^\infty)} \simeq \|g\|_{L^p(\mathbb{T}^\infty)}$ . This holds uniformly in  $\epsilon$ . The corollary now follows by averaging over  $\epsilon$  and invoking Khintchine's inequality [5, p. 435].  $\square$

This result should be compared to a Paley-Littlewood inequality obtained from martingale theory. Indeed, a function  $f \in L^p(\mathbb{T}^\infty)$  may be considered as a martingale  $\{f_{(N)}\}$  with respect to the filtration induced by the increasing sequence of  $\sigma$ -algebras corresponding to the sequence  $\{\mathbb{T}^N\}_{N \in \mathbb{N}}$ . The function  $f_{(N)}$ , also called the conditional expectation, is obtained from  $f$  by integrating away all but the  $N$  first variables (see, e.g., [8] where these are called the ' $N$ :te Abschnitt'). A Paley-Littlewood inequality is now obtained as a direct corollary of the classical Burkholder's square function inequality [3] (see also [5, Theorem 5.4.7]). Set  $\Delta_N f = f_N - f_{N-1}$ . Then

$$\|f\|_{L^p(\mathbb{T}^\infty)} \simeq \|(\sum |\Delta_N f|^2)^{1/2}\|_{L^p(\mathbb{T}^\infty)}. \quad (8)$$

Actually, the same argument that was used to prove Corollary 3 yields (8) without using probability theory (this observation was applied in [1]).

In the following corollary, we consider the functions  $1, 2^{-s}, 3^{-s}, \dots$ . It is clear that they form an orthogonal basis in  $\mathcal{H}^2$ . Luckily, they also yield a natural basis in  $\mathcal{H}^p$ :

**Corollary 4.** *Suppose  $p \in (1, \infty)$ . Then the functions  $n^{-s}$ ,  $n = 1, 2, \dots$ , form a Schauder basis for  $\mathcal{H}^p$ .*

*Proof.* By the density and independence of these functions, and standard Schauder basis theory, it suffices to establish that the truncations  $\sum_{n=1}^{\infty} a_n n^{-s} \mapsto \sum_{n=1}^N a_n n^{-s}$  are bounded on  $\mathcal{H}^p$ , uniformly with respect to  $N$ . Let  $\alpha \in (0, 1/2)$  be an irrational number. According to Corollary 1, the indicator functions of the intervals  $(0, N + \alpha)$  yield uniformly bounded multipliers on  $L^p(\mathbb{T}^\infty)$ . The result follows.  $\square$

Although we have not been able to find this result stated explicitly in the literature, we indicate how it can be deduced from [10, Theorem 8.7.2]. This result deals with the space  $L^p(G)$ , where  $G$  is a compact abelian group that has a dual  $\Gamma$  which admits an order relation  $P$ . I.e.,  $P$  is a subset of  $\Gamma$  such that

$$P \cup (-P) = \Gamma, \quad \text{and} \quad P \cap (-P) = \{0\}.$$

Under any such order relation one can define  $\text{sgn}(\gamma) \in \{-1, 0, 1\}$  according to whether or not  $\gamma$  is in  $P$  or is in  $\{0\}$ . With this, the statement is that the Hilbert transform

$$T_P : \sum_{\gamma \in \Gamma} a_\gamma e_\gamma \mapsto -i \sum_{\gamma \in \Gamma} \text{sgn}(\gamma) e_\gamma$$

is bounded on  $L^p(G)$ , where  $e_\gamma$  is the Fourier character corresponding to  $\gamma \in \Gamma$ . In particular,  $P = \{\nu : \log r_\nu \leq 0\}$  is an order relation in the dual  $\mathbb{Z}_{\text{fin}}^\infty$  of  $\mathbb{T}^\infty$ . Hence the corresponding Riesz projection  $R_P$ , where  $R_P e_\gamma := \chi_{\{r_\gamma \geq 0\}} e_\gamma$ , is bounded on  $L^p(\mathbb{T}^\infty)$ . If  $r \rightarrow \nu(r)$  is the inverse of the map  $r$ , we obtain uniformly in  $N$

$$\left\| \sum_{n \leq N} a_n n^{-s} \right\|_{\mathcal{H}^p} = \left\| e_{\nu(N)} R_P(e_{-\nu(N)} f) \right\|$$

for functions  $f(s) = \sum_{n \in \mathbb{N}} a_n n^{-s}$  in  $\mathcal{H}^p$ . As above, it follows immediately that  $\{n^{-s}\}$  is a Schauder basis for  $\mathcal{H}^p$  when  $p > 1$ .

We end with the following open questions which may seem innocent, but they could be somewhat hard taking into account the quite intractable and mysterious nature of the spaces  $\mathcal{H}^p$  for  $p \neq 2$  as discussed, e.g., in [11].

**Question 1.** Does  $\mathcal{H}^1$  have a Schauder basis?

**Question 2.** Does  $\mathcal{H}^p$  have an unconditional basis if  $p \in (1, \infty) \setminus \{2\}$ ?

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